Int. J. Heat Mass Transfer. Vol. 14, pp. 1759–1770. Pergamon Press 1971. Printed in Great Britain.

AN EXTENDED VARIATIONAL FORMULATION OF THE NON-LINEAR HEAT AND MASS TRANSFER IN A POROUS MEDIUM†

I. J. KUMAR
Defence Science Laboratory, Metcalfe House, Delhi-6, India

(*Received* 15 *December* 1969)

Abstract—A variational formulation based on the method of "local potential" is presented for the non-linear heat and mass transfer during drying of a moist body. A functional is obtained from which the balance equations of the process follow as Euler–Lagrange equations in the extended sense of Prigogine and Glansdorff. The method has been applied to the problem of heat and mass transfer in an infinite plate with temperature and moisture dependent thermal and mass diffusivities under boundary conditions of the first kind.

NOMENCLATURE

x_i,	Cartesian length coordinates, $i = 1, 2, 3$ [m];
t,	temperature [°C];
θ,	moisture transfer potential [°M];
τ,	time [h];
a_q,	thermal diffusivity coefficient [m^2/h];
a_m,	diffusion coefficient of moisture in capillary-porous body [m^2/h];
ε,	coefficient of internal moisture evaporation;
ρ,	specific heat of evaporation [kcal/kg];
δ_1,	thermal gradient coefficient [1/°C];
C_m,	specific isothermal mass capacity of moist body [kg/kg °M];
C_q,	specific heat capacity of moist body [kcal/kg °C];
λ_q,	thermal conductivity [kcal/mh °C];
λ_m,	moisture conductivity ($= a_m C_m \gamma$) [kg/mh°M];
$\Theta^*(X_i, Fo)$,	macroscopic mass-transfer potential distribution;
γ,	density of the porous medium [kg/m^3];
L,	half thickness of the infinite plate;
Lu,	the Luikov number ($= a_m/a_q$);
Ko,	the Kossovich number ($= \rho c_m \Delta\theta / c_q t$);
Ko^*,	modified Kossovich number ($= \varepsilon Ko$);
Pn,	the Posnov number ($= \delta_1 \Delta t / c_m \Delta\theta$);
Fo,	Fourier number;
T	dimensionless temperature: $(t - t_0)/t_0$;
Θ,	dimensionless mass transfer potential $(1 - \theta/\theta_0)$;
X_i,	dimensionless Cartesian coordinate ($= x_i/L$);
$T^*(X_i, Fo)$,	macroscopic temperature distribution;
σ_1,	constant specifying the depen-

† A major part of the work on this problem was completed when the author was at Heat and Mass Transfer Institute, Minsk, USSR, on an invitation of the USSR Academy of Sciences which he thankfully acknowledges.

dence of a_q on T, equation (3.1);

σ_2, constant specifying the dependence of a_m on T, equation (3.2);

Subscript 0 refers to initial values.

INTRODUCTION

THE VARIATIONAL formulation of non-linear problems in transport processes is comparatively of recent origin. Such a formulation provides the engineers with direct methods of calculus of variations for obtaining approximate analytic solutions to the problems which are otherwise tractable only by elaborate numerical techniques. The variational formulations of such problems start with the construction of a functional called "the Lagrangion" from which the balance equations follow as the Euler–Lagrange equations for the extremum of the Lagrangian. The first step in the direction of obtaining the Lagrangian was taken with the enunciation of the theorem of Minimum Entropy Production [1]. This theorem however could be held valid only under the following restrictive assumptions of

(a) linear phenomenolgical laws,
(b) symmetry of the matrix of phenomenological coefficients,
(c) absence of mechanical dissipation of energy.

Under the above restrictions, entropy production could be used as the evolution criterion for only linear processes for which exact methods of solution already existed. Prigogine and Glansdorff [2, 3] have given a generalized evolution criterion valid for any system without restrictions of the type stated above from which the balance equations of energy, momentum and mass follow as the Euler–Lagrange equations. This extended variational formulation is based on "local potential" which has a specialized construction. Local potential is a function of the dependent variables in the usual form which are subject to variation. It is also a function of dependent variables at a stationary state which are not subject to variation. The balance equations are recovered from the functional by obtaining the variation with respect to each of the variables, equating it to zero and then revoking the additional condition that values of the variables in the varied state are

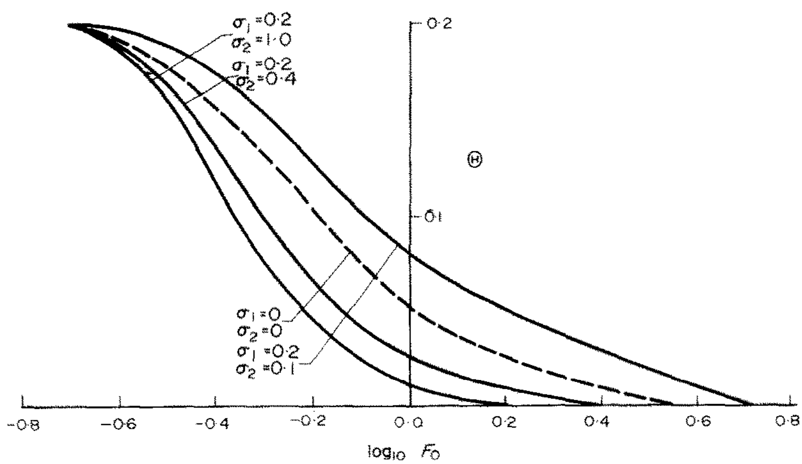

FIG. 1. Effect of variability of a_m, a_q on Θ (0, Fo) for $Lu_0 = 1$ $Ko^*Pn = 0{\cdot}2$ $Ko^* = 0{\cdot}5$ $\sigma_1 = 0; 0{\cdot}2$ $\sigma_2 = 0; 0{\cdot}1; 0{\cdot}4; 1$.

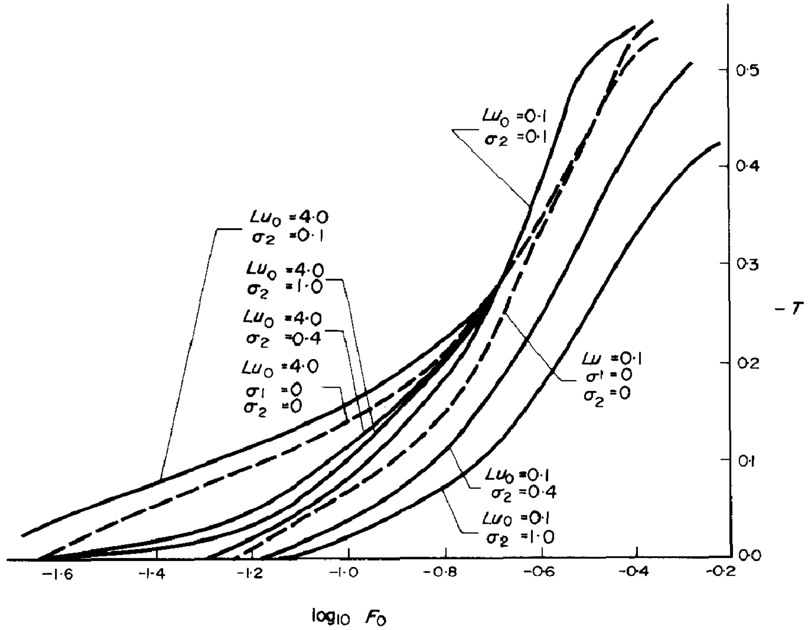

FIG. 2. Effect of variability of a_m, a_q on $T(0, Fo)$ for $Lu_0 = 0{\cdot}1, 4$ $Ko^*Pn = 0{\cdot}2$ $\sigma_1 = 0{\cdot}2$; 0 $\sigma_2 = 0; 0{\cdot}1; 0{\cdot}4; 1$.

equal to their value in the stationary state. The local potential formulation has recently been extended [4, 5] to transient processes and applied by Hays [6] and Curd [7, 8] to the solution of various non-linear problems in heat-conduction as also of mass diffusion.

The purpose of this paper is to apply the extended variational formulation to the problem of combined heat and mass transfer in a porous medium with very general dependence of the thermophysical characteristics on the temperature and mass transfer potential. Integral methods for the non-linear heat and mass transfer based on the boundary-layer approach developed earlier [9] and [10] are restricted by the fact that firstly such an approach can be successfully used only in one-dimensional process and secondly only a very restricted class of dependences of thermophysical properties on the heat and mass transfer potentials could be taken into account. The present method is not faced with any of such restrictions.

First, "a local potential" for non-linear heat and mass transfer has been set up. For illustration the method is applied to the heat and mass transfer in an infinite plate of finite thickness. The effect of non-linearity due to variable thermophysical characteristics has been brought out by a comparative study in Figs. 1 and 2 of Θ and T in the non-linear problem studied here and the corresponding linear problem studied by Luikov and Mikhailov [11]. The effect of variability of the a_m and a_q for $Lu_0 < 1$ are found to be opposite to the corresponding effects for $Lu_0 > 1$.

2. FORMATION OF THE EXTENDED VARIATIONAL PRINCIPLE BASED ON LOCAL POTENTIAL

The equations of heat and mass transfer in a porous medium with thermophysical characteristics dependent on temperature and moisture transfer potentials are stated as

$$C_q\gamma\frac{\partial t}{\partial \tau}=\frac{\partial}{\partial x_i}\left(\lambda_q\frac{\partial t}{\partial x_i}\right)+\varepsilon p\gamma\frac{C_m}{C_q}\frac{\partial\theta}{\partial\tau}\cdots \quad (2.1)$$

$$C_m\gamma\frac{\partial\theta}{\partial\tau}=\frac{\partial}{\partial x_i}\left(\lambda_m\frac{\partial\theta}{\partial x_i}\right)+\frac{\partial}{\partial x_i}\left(\delta_1\lambda_m\frac{\partial t}{\partial x_i}\right) \quad (2.2)$$

$$i=1,2,3.$$

If the dependence of a_m, a_q, ε, δ_1 on temperature and moisture transfer potentials is defined by the following general relations:

$$\begin{aligned} a_q &= a_{q_0}F_1(t,\theta) & \varepsilon &= \varepsilon_0 F_2(t,\theta) \\ a_m &= a_{m_0}F_3(t,\theta) & \delta_1 &= \delta_0 F_4(t,\theta). \end{aligned} \quad (2.3)$$

On introducing the non-dimensional parameters defined in the nomenclature list we can write

$$\frac{\partial T}{\partial F_0}=\frac{\partial}{\partial X_i}\left[F_1(T,\Theta)\frac{\partial T}{\partial X_i}\right]-Ko^*F_2(T,\Theta)\times\frac{\partial\Theta}{\partial F_0} \quad (2.4)$$

$$\frac{\partial\Theta}{\partial F_0}=Lu_0\frac{\partial}{\partial X_i}\left[F_3(T,\Theta)\frac{\partial\Theta}{\partial X_i}\right]-Lu_0Pn_0\frac{\partial}{\partial X_i}\left[F_3F_4\frac{\partial T}{\partial X_i}\right] \quad (2.5)$$

where it has been assumed that C_m, C_q and γ are constant.

Now consider an isotopic capillary porous solid with volume V and surface S. The dimensionless temperature T and moisture-potential Θ can be considered to be composed of the macroscopic distribution $T^*(X_i, Fo)$ and $\Theta^*(X_i, Fo)$ and the arbitrary variation of δT and $\delta\Theta$ around the corresponding macroscopic distribution T^* and Θ^*. Thus we may write

$$\left.\begin{aligned} T(X_i,F_0) &= T^*(X_i,F_0)+\delta T(X_i,Fo) \\ \Theta(X_i,Fo) &= \Theta^*(X_i,Fo)+\delta\Theta(X_i,Fo) \end{aligned}\right] \quad (2.6)$$

and also

$$\begin{aligned} F_n(T,\Theta) &= F_n[T^*+\delta T,\Theta^*+\delta\Theta] \\ &= F_n(T^*,\Theta^*)+\delta F_n \quad n=1,2,3,4. \end{aligned} \quad (2.7)$$

Multiplying both sides of (2.4) and (2.5) by $-\delta T$ and $-\delta\Theta$, respectively, we get

$$-\frac{\partial T}{\partial F_0}\delta T=-\frac{\partial}{\partial X_i}\left[F_1\frac{\partial T}{\partial X_i}\right]\delta T+Ko^*F_2\frac{\partial\Theta}{\partial Fo}\delta T \quad (2.8)$$

and

$$-\frac{\partial\Theta}{\partial F_0}\delta\Theta=-Lu_0\frac{\partial}{\partial X_i}\left[F_3\frac{\partial\Theta}{\partial X_i}\right]\delta\Theta+Lu_0Pn_0\frac{\partial}{\partial X_i}\left[F_3F_4\frac{\partial T}{\partial X_i}\right] \quad (2.9)$$

or using (2.6)

$$-\frac{1}{2}\frac{\partial}{\partial F_0}(\delta T)^2=\frac{\partial T^*}{\partial F_0}\delta T-\frac{\partial T}{\partial X_i}\left[F_1\frac{\partial T}{\partial X}\delta T\right]+Ko^*F_2\frac{\partial\Theta}{\partial F_0}\delta T \quad (2.10)$$

and

$$-\frac{1}{2}\frac{\partial}{\partial F_0}(\delta\Theta)^2=\frac{\partial\Theta^*}{\partial F_0}\delta\Theta-Lu_0\frac{\partial}{\partial X_i}\times\left[F_3\frac{\partial\Theta}{\partial X_i}\delta\Theta\right]+Lu_0Pn\frac{\partial}{\partial X_i}\times\left[F_3F_4\frac{\partial T}{\partial X_i}\delta\Theta\right]. \quad (2.11)$$

Integrating both sides of (2.10) and (2.11) over the volume V and also over Fo (non-dimensional time), using the divergence theorem

$$-\frac{1}{2}\int(\delta T)^2\,\mathrm{d}v=\int_v\int_{Fo}\left[\frac{\partial T^*}{\partial Fo}\delta T+Ko^*F_2\times\frac{\partial\Theta}{\partial Fo}\delta T+\frac{1}{2}F_1\delta\left(\frac{\partial T}{\partial X_i}\right)^2\right]\mathrm{d}v\,\mathrm{d}Fo-\int_S\int_{Fo}F_1\frac{\partial T}{\partial X_i}\delta Tn\,\mathrm{d}s\,\mathrm{d}Fo \quad (2.12)$$

$$-\frac{1}{2}\int_v(\delta\Theta)^2\,\mathrm{d}v=\int_v\int_{Fo}\frac{\partial\Theta^*}{\partial Fo}\delta\Theta+\frac{1}{2}Lu_0F_3$$

$$\delta\left(\frac{\partial \Theta}{\partial X_i}\right)^2 - Lu_0 Pn_0\left(F_3F_4\frac{\partial T^*}{\partial X_i}\frac{\partial(\delta\Theta)}{\partial X_i}\right) \mathrm{d}v\,\mathrm{d}Fo$$

$$-\int_s\int_{Fo} Lu_0F_3\frac{\partial \Theta}{\partial X_i}n\delta\Theta\,\mathrm{d}s\,\mathrm{d}Fo + \int_s\int_{Fo} Lu_0Pn_0F_3F_4 \times \frac{\partial T^*}{\partial X_i}n\delta\Theta\,\mathrm{d}s\,\mathrm{d}F_o. \quad (2.13)$$

The surface integrals on the right-hand side of equation (2.12) and (2.13) vanish for either conditions; T and Θ being constant at the surface or the surface being insulated to heat and mass transfer. On substituting (2.7) on the right-hand side of equations (2.12) and (2.13) and neglecting the terms $\delta T \ll T^*$ and $\delta\,\Theta \ll \Theta^*$, we get to the first approximation

$$\delta J_1 \doteq -\frac{1}{2}\int_v(\delta T)^2\,\mathrm{d}v = \int_v\int_{Fo}\left[\frac{\partial T^*}{\partial Fo}\delta T + Ko^*F_2^* \times \frac{\partial \Theta}{\partial Fo} + \frac{1}{2}F_1^*\delta\left(\frac{\partial T}{\partial X_i}\right)^2\right]\mathrm{d}v\,\mathrm{d}Fo \quad (2.14)$$

and

$$\delta J_2 = -\frac{1}{2}\int_v(\delta\Theta)^2\,\mathrm{d}v = \int_v\int_{Fo}\left[\frac{\partial \Theta^*}{\partial Fo}\delta\Theta + \frac{1}{2}Lu_0F_3{}^*\delta\left(\frac{\partial \Theta}{\partial X_i}\right)^2 - Lu_0Pn_0\left[F_3^*F_4^* \times \frac{\partial T^*}{\partial X_i}\delta\left(\frac{\partial \Theta}{\partial X_i}\right)\right]\right]\mathrm{d}v\,\mathrm{d}Fo. \quad (2.15)$$

In view of the fact that the left-hand side of the above equations are negative and definite, we get

$$\delta J = \delta J_1 + \delta J_2 \leqslant 0. \quad (2.16)$$

Consequently, the required macroscopic temperature distributions T^* and moisture transfer potential Θ^* is characterised by the extremum conditions

$$\left(\frac{\delta J}{\delta T}\right)_{T^*} = 0, \qquad \left(\frac{\delta J}{\delta \Theta}\right)_{\Theta^*} = 0 \quad (2.17)$$

with subsidiary conditions

$$T = T^*$$
$$\Theta = \Theta^* \quad (2.18)$$

where

$$J = \int_v\int_{Fo}\left[\frac{\partial T^*}{\partial Fo}T + Ko^*F_2^*\frac{\partial \Theta^*}{\partial Fo}T + \frac{1}{2}F_1^*\left(\frac{\partial T}{\partial X_i}\right)^2 + \Theta\frac{\partial \Theta^*}{\partial X_i} + \frac{1}{2}Lu_0F_3^*\left(\frac{\partial \Theta}{\partial X_i}\right)^2 - Lu_0Pn_0F_3^*F_4^* \times \frac{\partial T^*}{\partial X_i}\frac{\partial \Theta}{\partial X_i}\right]\mathrm{d}v\,\mathrm{d}Fo. \quad (2.19)$$

It may be noted that in the above extended variational principle based on local potential the thermo-physical parameters a_m, a_q, ε and δ_1 can be taken to be any arbitrary function of θ and t provided integrations with respect to V and Fo can be carried out for the assumed profiles for T, T^*, Θ and Θ^*. The stipulation of specific type of boundary conditions namely, the constancy of Θ and T or vanishing of

$$\frac{\partial \Theta}{\partial X_i}, \frac{\partial T}{\partial X_i},$$

the normal derivatives at the boundary, is not a serious limitation. While in quite a number of problems these conditions are actually satisfied, in others pertaining to one-dimensional heat and mass transfer, the surface integrals can be easily evaluated for the boundary conditions of the second and third kind.

3. APPLICATION TO NON-LINEAR HEAT AND MASS TRANSFER IN AN INFINITE PLATE UNDER BOUNDARY CONDITIONS OF THE FIRST KIND

Consider the heat and mass transfer in a moist inifinite flat plate $-1 \leqslant x \leqslant 1$ under the boundary condition that there is no mass-loss from the boundaries while the temperature at the boundaries is kept constant. We also assume the following dependence of thermo-physical properties

$$a_q = a_{q0}(1 + \sigma_1 T) \quad (3.1)$$

$$a_m = a_{m_0}(1 + \sigma_2 \Theta) \tag{3.2}$$

the other thermophysical properties being assumed constant. The process can be mathematically stated by the following equations:

$$\frac{\partial T}{\partial Fo} = \frac{\partial}{\partial X}\left[(1 + \sigma_1 T)\frac{\partial T}{\partial X}\right] - Ko^* \frac{\partial \Theta}{\partial Fo} \tag{3.3}$$

and

$$\frac{\partial \Theta}{\partial Fo} = \frac{\partial}{\partial X}\left[Lu_0(1 + \sigma_2 \Theta)\frac{\partial \Theta}{\partial X}\right] - Lu_0 Pn_0 \frac{\partial}{\partial X}\left[(1 + \sigma_2 \Theta)\frac{\partial T}{\partial X}\right]$$

$$0 < X < 1 \qquad Fo > 0 \tag{3.4}$$

with the initial conditions

$$\Theta(x, 0) = 0 \qquad T(x, 0) = 0 \tag{3.5}$$

and the boundary conditions

$$\int_0^1 \Theta(x, Fo)\,\mathrm{d}X = 0 \qquad T(1, Fo) = -1 \tag{3.6}$$

and the condition due to symmetry

$$\frac{\partial T(0, Fo)}{\partial X} = \frac{\partial \Theta(0, Fo)}{\partial X} = 0. \tag{3.7}$$

The first of conditions (3.6) means that the total moisture content of the plate remains constant.

The functional $J(T, T^*, \Theta, \Theta^*)$ whose variation with respect to T and Θ equated to zero along with the subsidiary condition (2.18) is equivalent to the equations (3.3) and (3.4) is given as a particular case of equation (2.19). Thus,

$$J(T, T^*, \Theta, \Theta^*) = \int_0^1 \int_{Fo} \left[\begin{array}{l} T\dfrac{\partial T^*}{\partial Fo} + Ko^* \dfrac{\partial \Theta}{\partial Fo} T + \dfrac{(1 + \sigma_1 T^*)}{2}\left(\dfrac{\partial T}{\partial X}\right)^2 + \Theta \dfrac{\partial \Theta^*}{\partial Fo} \\ + \dfrac{Lu_0}{2}(1 + \sigma \Theta^*)\left(\dfrac{\partial \Theta}{\partial X}\right)^2 - Lu_0 Pn_0(1 + \sigma_2 \Theta^*)\dfrac{\partial T^*}{\partial X}\dfrac{\partial \Theta}{\partial X} \end{array}\right] \times \mathrm{d}X\mathrm{d}Fo. \tag{3.8}$$

The surface integrals vanish because of the boundary conditions (3.6) and (3.7). To find the solution of the problem, we use the self-consistent method set forth by Prigogine and Glansdorff [3,5]. We have to choose a profile for T and Θ satisfying the initial and boundary conditions and containing some undetermined coefficients which would be determined so as to minimize J. Here we choose a profile which is similar to the exact solution of the linear problem studied in [11] ($\sigma_1 = 0, \sigma_2 = 0$) and satisfies the initial and boundary conditions (3.5) and (3.6). We assume

$$T(X, Fo) = -\left[1 + \sum_{n=1}^{\infty} \sum_{i=1}^{2} c_{ni} \cos(v_i \mu_n X) \exp(-\mu_n^2 Fo)\right] \tag{3.9}$$

$$\Theta(X, Fo) = \frac{1}{Ko^*} \sum_{n=1}^{\infty} \sum_{i=1}^{2} c_{ni}(1 - v_i^2) \cos(v_i \mu_n X) \exp(-\mu_n^2 Fo) \tag{3.10}$$

where

$$c_{n1} = 2v_1(1 - v_2^2)\sin(v_2\mu_n) \Big/ \left[\mu_n \sin v_1\mu_n \sin v_2\mu_n - \frac{\mu_n}{Lu_0}\cos v_1\mu_n \cos v_2\mu_n\right] \tag{3.11}$$

$$c_{n2} = 2v_2(1 - v_1^2)\sin(v_1\mu_n) \Big/ \left[\mu_n \sin v_1\mu_n \sin v_2\mu_n - \frac{\mu_n}{Lu_0}\cos v_1\mu_n \cos v_2\mu_n\right] \tag{3.12}$$

and v_1 and v_2 are given by

$$v_i^2 = \frac{1}{2}\left[\left(1 + Ko^* Pn_0 + \frac{1}{Lu_0}\right) + (-1)^i \Big/\right.$$

$$\left[\left(1 + Ko^*Pn_0 + \frac{1}{Lu_0}\right)^2 - \frac{4}{Lu_0}\right]\Bigg]_{i=1,2}. \quad (3.13)$$

Also

$$T^*(X, Fo) = -\left[1 + \sum_{n=1}^{\infty}\sum_{i=1}^{2} c_{n1}^* \cos(\nu_2\mu_n^* X)\exp(-\mu_n^{*2}Fo)\right] \quad (3.14)$$

$$\Theta^*(X, Fo) = \frac{1}{Ko^*}\sum_{n=1}^{\infty}\sum_{i=1}^{2} c_{n1}^*(1 - \nu_i^2)\cos(\nu_i\mu_n^* X)\exp(-\mu_n^{*2}Fo) \quad (3.15)$$

with corresponding values of C_{n1}^* and C_{n2}^*.

The unknown coefficients μ_n in the profiles (3.9) and (3.10) are to be determined from the conditions

$$\partial J/\partial\mu_n = 0 \quad (3.16)$$

with the conditions of self-consistency

$$\mu_n = \mu_n^*. \quad (3.17)$$

The assumption of the profiles (3.9) and (3.10) offer some distinct advantages. Firstly, it can be easily verified that at $\sigma_1 = 0$ and $\sigma_2 = 0$ the solutions obtained from the variational method is the same as the exact solution for the linear case and secondly, we can now obtain T and $\sigma_2 = 0$ only by determination of one set of coefficients μ_n. Substituting T, T^*, Θ and Θ^* from the equations (3.9), (3.10), (3.13) and (3.14) on the right-hand side of equation (3.16), finding $\partial J/\partial\mu_n$, taking μ_n^* to be constant, equating $\partial J/\partial\mu_n$ to zero and then imposing the self-consistency condition (3.17), we get after algebraic simplification the following infinite system of non-linear algebraic equations

$$\sum_{m=1}^{\infty}\sum_{i=1}^{2}\sum_{J=1}^{2}\left[\begin{array}{l}\dfrac{B_{mJ}\nu_i\nu_J}{\mu_n^2 + \mu_m^2}\left[1 - \sigma_1 - \dfrac{Lu_0(1 - \nu_J^2)}{Ko^*}\left(\dfrac{1 - \nu_1^2}{Ko^*} + Pn_0\right)\right] \\ \left[C'_{ni}\beta_{ij}^{ss}\mu_n + \dfrac{B_{ni}\beta_{ij}^{ss}(\mu_m^2 - \mu_n^2)}{\mu_m^2 - \mu_n^2} + B_{ni}\mu_n\nu_i\beta_{ij}^{es}\right] \\ + \dfrac{\mu_m B_{mJ}}{\mu_m^2 + \mu_n^2}\left[\nu_J^2 - \dfrac{(1 - \nu_1^2(1 - \nu_J^2)}{Ko^*}\right]\left[-C'_{ni}\beta_{iJ}^{cc} + B_{ni}\beta_{iJ}^{sc} + \dfrac{2B_{ni}\mu_n\beta_{iJ}^{cc}}{\mu_m^2 + \mu_n^2}\right]\end{array}\right]$$

$$+ \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\sum_{i=1}^{2}\sum_{J=1}^{2}\sum_{l=1}^{2}\frac{C_{kl}B_{mJ}\nu_i\nu_J}{\mu_m^2 + \mu_n^2 + \mu_k^2} \times \left[\begin{array}{l} - \sigma_1 - \dfrac{Lu_0\sigma_2(1 - \nu_2^2)(1 - \nu_J^2)(1 - \nu_l^2)}{Ko^{*3}} \\ - \dfrac{Lu_0 Pn_0\sigma_2(1 - \nu_2^2)(1 - \nu_l^2)}{Ko^{*2}}\end{array}\right]$$

$$[C'_{ni}\mu_n\beta_{iJl}^{ssc} + B_{ni}\nu_i\beta_{iJ}^{cs}\beta_{iJl}^{csc} + B_{ni}\beta_{iJl}^{ssc} + B_{ni}\beta_{ijl}^{sc}(\mu_m^2 + \mu_n^2 + \mu_k^2)] = 0 \quad (3.18)$$

where

$$B_{n1} = \nu_1(1 - \nu_2^2)\sin\nu_2\mu_n \quad (3.19)$$

$$B_{n2} = \nu_2(1 - \nu_1^2)\sin\nu_1\mu_n \quad (3.20)$$

$$B_{m1} = \nu_1(1 - \nu_2^2)\sin\nu_2\mu_m \quad (3.21)$$

$$B_{m2} = \nu_2(1 - \nu_1^2)\sin\nu_1\mu_m. \quad (3.22)$$

C_{ni} are given by equations (3.11) and (3.12) and

$$C'_{ni} = \frac{\partial C_{ni}}{\partial \mu_n} \tag{3.23}$$

and if

$$\alpha_{lk} = \nu_l \mu_k. \tag{3.24}$$

Then

$$\beta^{ss}_{iJ} = \int_0^1 \sin(\nu_i \mu_n X) \sin(\nu_J \mu_m X)\, dX$$

$$= \frac{\alpha_{Jm} \sin\alpha_{in} \cos\alpha_{Jm} - \alpha_{in} \cos\alpha_{in} \sin\alpha_{Jm}}{\alpha_{in}^2 - \alpha_{Jm}^2}$$

$$= \frac{1}{2} - \frac{\sin 2\alpha_{in}}{4} \qquad \begin{matrix} i = J \\ m = n \end{matrix} \tag{3.25}$$

$$\beta^{cc}_{iJ} = \int_0^1 \cos(\nu_i \mu_n X) \cos(\nu_J \mu_m X)\, dX$$

$$= \frac{\alpha_{in} \sin\alpha_{in} \cos\alpha_{Jm} - \alpha_{Jm}}{\alpha_{in}^2 - \alpha_{Jm}^2} \cos\alpha_{in} \sin\alpha_{Jm} \qquad \begin{matrix} i \neq J \\ m \neq n \end{matrix}$$

$$= \frac{1}{2} + \frac{\sin 2\alpha_{in}}{4\alpha_{in}} \qquad \begin{matrix} i = J \\ m = n \end{matrix} \tag{3.26}$$

$$\beta^{sc}_{iJ} = \int_0^1 X \sin(\nu_i \mu_n X) \cos(\nu_J \mu_m X)\, dX$$

$$= \frac{\alpha_{in} \cos\alpha_{in} \cos\alpha_{Jm} + \alpha_{Jm} \sin\alpha_{in} \sin\alpha_{Jm}}{\alpha_{Jm}^2 - \alpha_{in}^2}$$

$$+ \frac{(\alpha_{in}^2 + \alpha_{Jm}^2) \sin\alpha_{in} \cos\alpha_{Jm} - 2\alpha_{in}\alpha_{Jm} \cos\alpha_{in} \sin\alpha_{Jm}}{(\alpha_{in}^2 - \alpha_{Jm}^2)^2} \qquad \begin{matrix} i \neq J \\ m \neq n \end{matrix}$$

$$= \frac{\sin 2\alpha_{in}}{8\alpha_{in}^2} - \frac{\cos 2\alpha_{in}}{4\alpha_{in}} \qquad \begin{matrix} i = J \\ m = n \end{matrix} \tag{3.27}$$

$$\beta^{cs}_{iJ} = \int_0^1 X \cos(\nu_i \mu_n X) \sin(\nu_J \mu_m X)\, dX$$

$$= \frac{\alpha_{in} \sin\alpha_{in} \sin\alpha_{Jm} + \alpha_{Jm} \cos\alpha_{in} \cos\alpha_{Jm}}{\alpha_{in}^2 - \alpha_{Jm}^2}$$

$$+ \frac{(\alpha_{in}^2 + \alpha_{Jm}^2) \cos\alpha_{in} \sin\alpha_{Jm} - 2\alpha_{in}\alpha_{Jm} \cos\alpha_{Jm} \sin\alpha_{in}}{(\alpha_{in}^2 - \alpha_{Jm}^2)^2} \qquad \begin{matrix} i \neq J \\ m \neq n \end{matrix} \tag{3.28}$$

$$= \frac{1}{2}\left[\frac{\sin 2\alpha_{in}}{4\alpha_{in}^2} - \frac{\cos 2\alpha_{in}}{2\alpha_{in}}\right] \qquad \begin{matrix} i = J \\ m = n \end{matrix}$$

$$\beta^{ssc}_{iJl} = \int_0^1 \sin(v_i\mu_n X)\sin(v_J\mu_m X)\cos(v_l\mu_k X)\,dX$$

$$= \cos\alpha_{lk}(\alpha_{Jm}\sin\alpha_{in}\cos\alpha_{Jm} - \alpha_{in}\cos\alpha_{in}\sin\alpha_{Jm})/(\alpha_{in}^2 - \alpha_{Jm}^2)$$

$$+ \frac{\alpha_{lk}}{2}\left[\frac{(\alpha_{in}-\alpha_{Jm})\sin\alpha_{lk}\cos(\alpha_{in}-\alpha_{Jm}) - \alpha_{lk}\cos\alpha_{lk}\sin(\alpha_{in}-\alpha_{Jm})}{(\alpha_{in}-\alpha_{Jm})[\alpha_{lk}^2 - (\alpha_{in}-\alpha_{Jm})^2]} + \frac{\alpha_{lk}\cos\alpha_{lk}\sin(\alpha_{in}+\alpha_{Jm}) - (\alpha_{in}+\alpha_{Jm})\sin\alpha_{lk}\cos(\alpha_{in}+\alpha_{Jm})}{(\alpha_{in}+\alpha_{Jm})[\alpha_{lk}^2 - (\alpha_{in}+\alpha_{Jm})^2]}\right] \quad \begin{matrix} i \neq J \neq l \\ m \neq n \neq k \end{matrix}$$

$$= \frac{\sin\alpha_{in}}{4\alpha_{in}} - \frac{\sin 3\alpha_{in}}{12\alpha_{in}} \quad \begin{matrix} i = J = l \\ m = n = k \end{matrix}$$

$$= \frac{\sin\alpha_{lk}}{2\alpha_{lk}} - \frac{1}{2}\left[\frac{2\alpha_{lk}\sin 2\alpha_{in}\cos\alpha_{lk} - \alpha_{lk}\sin\alpha_{lk}\cos 2\alpha_{in}}{4\alpha_{in}^2 - \alpha_{lk}^2}\right] \quad \begin{matrix} i = J \neq l \\ m = n \neq k \end{matrix} \tag{3.29}$$

$$\beta^{csc}_{iJl} = \int_0^1 X\cos(v_i\mu_n X)\sin(v_J\mu_m X)\cos(v_l\mu_k X)\,dX$$

$$= \cos\alpha_{lk}\left[\frac{\alpha_{in}\sin\alpha_{in}\sin\alpha_{Jm} + \alpha_{Jm}\cos\alpha_{in}\cos\alpha_{Jm}}{\alpha_{in}^2 - \alpha_{Jm}^2} + \frac{(\alpha_{in}^2 + \alpha_{Jm}^2)\cos\alpha_{in}\sin\alpha_{Jm} - 2\alpha_{in}\alpha_{Jm}\sin\alpha_{in}\cos\alpha_{Jm}}{(\alpha_{in}^2 - \alpha_{Jm}^2)^2}\right]$$

$$+ \frac{\alpha_{lk}}{2(\alpha_{in}-\alpha_{Jm})^2[\alpha_{lk}^2 - (\alpha_{in}-\alpha_{Jm})^2]}$$

$$\times \left[\begin{aligned} &(\alpha_{in}-\alpha_{Jm})\alpha_{lk}\cos(\alpha_{in}-\alpha_{Jm})\left\{\cos\alpha_{lk} - \frac{\sin\alpha_{lk}[\alpha_{lk}^2 + (\alpha_{in}+\alpha_{Jm})^2]}{[\alpha_{lk}^2 - (\alpha_{in}-\alpha_{Jm})^2]}\right\} \\ &+ (\alpha_{in}-\alpha_{Jm})\sin(\alpha_{in}-\alpha_{Jm})\left\{\sin\alpha_{lk} + \frac{2\alpha_{lk}\cos\alpha_{lk}}{[\alpha_{lk}^2 - (\alpha_{in}-\alpha_{Jm})^2]}\right\} \\ &- \alpha_{lk}\cos\alpha_{lk}\sin(\alpha_{in}-\alpha_{Jm}) + (\alpha_{in}-\alpha_{Jm})\sin\alpha_{lk}\cos(\alpha_{in}-\alpha_{Jm}) \end{aligned}\right]$$

$$\frac{\alpha_{lk}}{2(\alpha_{in}+\alpha_{Jm})^2[\alpha_{lk}^2 - (\alpha_{in}+\alpha_{Jm})^2]}$$

$$\left[\begin{aligned} &(\alpha_{in}+\alpha_{Jm})\alpha_{lk}\cos(\alpha_{in}+\alpha_{Jm})\left\{\cos\alpha_{lk} - \frac{\sin\alpha_{lk}[\alpha_{lk}^2 + (\alpha_{in}+\alpha_{Jm}^2)]^2}{\alpha_{lk}[\alpha_{lk}^2 - (\alpha_{in}+\alpha_{Jm})^2]}\right\} \\ &+ (\alpha_{in}+\alpha_{Jm})^2\sin(\alpha_{in}+\alpha_{Jm})\left\{\sin\alpha_{lk} + \frac{2\alpha_{lk}\cos\alpha_{lk}}{[\alpha_{lk}^2 - (\alpha_{in}+\alpha_{Jm})^2]}\right\} \\ &+ (\alpha_{in}+\alpha_{Jm})\sin\alpha_{lk}\cos(\alpha_{in}+\alpha_{Jm}) - \alpha_{lk}\cos\alpha_{lk}\sin(\alpha_{in}+\alpha_{Jm}) \end{aligned}\right]$$

$$= \frac{\sin\alpha_{in}}{4\alpha_{in}^2} - \frac{\cos\alpha_{in}}{4\alpha_{in}} - \frac{\cos 3\alpha_{in}}{12\alpha_{in}} + \frac{\sin 3\alpha_{in}}{36\alpha_{in}^2} \quad \begin{matrix} i = J = l \\ m = n = k \end{matrix}$$

$$= \frac{1}{2}\left[\frac{\alpha_{lk}\sin 2\alpha_{in}\sin\alpha_{lk} + 2\alpha_{in}\cos 2\alpha_{in}\cos\alpha_{lk}}{4\alpha_{in}^2 - \alpha_{lk}^2}\right] + \frac{1}{2}\left[\frac{(4\alpha_{in}^2 + \alpha_{lk}^2)\cos 2\alpha_{in}\sin\alpha_{lk} - 4\alpha_{in}\alpha_{lk}\cos\alpha_{lk}\sin 2\alpha_{in}}{(4\alpha_{in}^2 - \alpha_{lk}^2)^2}\right] \quad \begin{matrix} i = J \neq l \\ m = n \neq k \end{matrix} \tag{3.30}$$

Equation (3.18) is a set of non-linear algebraic equations in terms of coefficients μ_n—when truncated at s term, these are of the form

$$F_i = (\mu_1, \mu_2, \ldots \mu_s) = 0$$

$$i = 1, 2, 3 \ldots s. \tag{3.31}$$

The solutions of these equations were obtained by Newton–Raphson's method [12]. The method was earlier applied by Hays [6,8] in the solution of a similar equation in case of heat conduction.

The calculations of roots of equation (3.18) were carried out on a computer for various values of σ_1 and σ_2 using as the starting values of μ_n the corresponding known values found in the solution of the corresponding linear problem [11] ($\sigma_1 = 0$, $\sigma_2 = 0$) and improving the values till the remainder is of order 10^{-4}. In this manner the first four roots of equation (3.18) were determined. These roots are tabulated in the following tables for various values of Ko^*, Pn and σ_1 and σ_2.

Table 1(a). First four roots of equation (3.18) $\sigma_1 = 0{\cdot}2$ $\sigma_2 = 1$

	$Ko^*Pn_0 = 0{\cdot}2$				$Ko^*Pn_0 = 0{\cdot}4$			
Lu_0	μ_{n1}	μ_{n2}	μ_3	μ_4	μ_1	μ_2	μ_3	μ_4
0·1	1·453	2·035	2·683	3·874	1·314	1·895	2·481	3·736
0·4	1·525	2·643	4·229	5·553	1·413	2·541	4·073	5·476
1·0	1·548	4·126	4·895	6·539	1·441	4·053	4·783	6·478
4·0	1·546	4·673	6·763	7·764	1·540	6·712	6·695	7·708

Table 1(b). Roots of equation (3.18) $\sigma_1 = 0{\cdot}2$ $\sigma_2 = 0{\cdot}4$

	$Ko^*Pn_0 = 0{\cdot}2$				$Ko^*Pn_0 = 0{\cdot}4$			
Lu_0	μ_1	μ_2	μ_3	μ_4	μ_1	μ_2	μ_3	μ_4
0·1	1·385	1·902	2·494	3·672	1·353	1·792	2·285	3·353
0·4	1·520	2·193	3·810	5·287	1·495	2·112	3·681	4·983
1·0	1·541	3·145	4·658	6·291	1·526	3·096	4·570	6·146
4·0	1·546	4·297	6·571	7·612	1·538	4·259	6·242	7·503

Table 1(c). Roots of equation (3.18) $\sigma_1 = 0{\cdot}2$ $\sigma_2 = 0{\cdot}1$

	$Ko^*Pn_0 = 0{\cdot}2$				$Ko^*Pn_0 = 0{\cdot}4$			
Lu_0	μ_1	μ_2	μ_3	μ_4	μ_1	μ_2	μ_3	μ_4
0·1	0·8013	1·325	1·584	2·456	0·7917	1·318	1·579	2·452
0·4	1·402	1·941	3·574	4·649	1·397	1·936	3·570	4·646
1·0	1·506	3·008	4·478	5·931	1·500	3·001	4·475	5·928
4·0	1·530	4·143	6·493	7·383	1·528	4·140	6·493	7·383

Table 2. Values of v_1 *and* v_2

	$Ko^*Pn_0 = 0{\cdot}2$		$Ko^*Pn_0 = 0{\cdot}4$	
Lu_0	v_1	v_2	v_1	v_2
0·1	0·9891	3·1971	0·9786	3·2314
0·4	0·9431	1·6764	0·8993	1·7582
1·0	0·8011	1·2483	0·7326	1·3650
4·0	0·4472	1·1180	0·4108	1·2171

RESULTS AND DISCUSSIONS

The effect of dependence of the moisture diffusivity a_m and thermal diffusivity a_q respectively on the mass transfer potential and the temperature is depicted in Figs. 1 and 2. In these figures the dotted curves represent the solutions of the linear problem for values of Lu_0 and Ko^*Pn_0, etc. as stated in these figures and have been plotted by numerical results from reference [11]. In Fig. 1 Θ (0, Fo), the non-dimensional mass-transfer potential at the symmetry surface is plotted as a function of non-dimensional time for four combinations of the variability characteristics namely ($\sigma_1 = 0$, $\sigma_2 = 0$), ($\sigma_1 = 0{\cdot}2$, $\sigma_2 = 0{\cdot}1; 0{\cdot}4; 1$) and $Lu_0 = 1$. It is clear from the figure that in general $\sigma_2 > \sigma_1$ makes the mass transfer faster. This is quite an expected result as in this case the effective Luikov number $[Lu_0(1 + \sigma_2\Theta)/(1 + \sigma_1 T)] > 1$. In this case the drying time is effectively reduced. In Fig. 2 $-T(0, Fo)$ is plotted against non-dimensional time for the above four combinations of σ_1 and σ_2 and for $Lu_0 = 0{\cdot}1$ and $Lu_0 = 4{\cdot}0$. It is seen that T is quite sensitive to the dependence of a_q on T for smaller values of Lu but it is not so for higher values. Further for $Lu_0 < 1$ and $\sigma_2 < \sigma_1$, the effect of non-linearity is more pronounced on T for large Fo while a similar effect is visible on T for small Fo at $Lu_0 > 1$ and $\sigma_2 > \sigma_1$. This can probably be explained by the fact that for $Lu_0 < 1$ mass transfer is slower than heat transfer and $\sigma_2 < \sigma_1$ only adds to this tendency with increasing time resulting in considerable slowing down the rate of change of Θ thus blocking the rate of increase of T due to coupling in the energy balance equation. For $Lu_0 < 1$ and $\sigma_2 > \sigma_1$ this effect has no definite tendency.

CONCLUSION

A variational principle based on "local potential" has been constructed for the simultaneous non-linear heat and mass transfer for a very general type of dependence of the thermodynamic characteristics of the medium on the temperature and mass transfer potential. The effect of variability of a_m and a_q on the temperature and moisture distributions in an infinite plate of finite thickness has been studied under boundary conditions of the first kind. It is seen that the method reduces to solution of a system of non-linear equations. The extension of the method to deal with more complex boundary conditions and also to the intensive drying problem (where apart from temperature and mass transfer, pressure distribution is also taken into account) would lead to some computational difficulties but these would be quite negligible in comparison to the elaborate and complex numerical procedures for solving a system of simultaneous non-linear partial differential equations.

ACKNOWLEDGEMENTS

The author is thankful to Dr R. R. Aggarwal, Assist. Director Maths, Defence Science Laboratory for his interest in the work and to the Director Defence Science Laboratory for permission to publish this paper.

REFERENCES

1. I. Prigogine, Moderation et transformations irreversibles des systems ouverts, *Bull. Acad. Belg. Cl. Sci.* **31**, 600 (1945).
2. P. Glansdorff and I. Prigogine, Sur les properties differentielles de la production d'entropie. *Physica's Grav.* **20**, 773 (1954).
3. P. Glansdorff and I. Prigogine, On a general evolution criterion in macroscopic physics, *Physica's Grav.* **30**, 351 (1964).
4. I. Prigogine and P. Glansdorff, Variational properties and fluctuation theory, *Physica* **31**, 1242 (1965).
5. P. Glansdorff, Mechanical flow processes and variational methods based on fluctuation theory, *Non-Equilibrium Thermodynamics, Variational Techniques and Stability,* edited by R. J. Donelly, R. Herman and I. Prigogine, pp. 45–54. University of Chicago Press, Chicago, Illinois (1966).
6. D. F. Hays, Variational formulation of heat equation: Temperature dependent thermal conductivity, *Non-Equilibrium Thermodynamics, Variational Techniques and Stability*, edited by R. J. Donelly, R. Herman and I. Prigogine, pp. 17–43, University of Chicago Press, Chicago, Illinois (1966).
7. D. F. Hays and H. N. Curd, Concentration dependent diffusion in a semi-infinite medium. *J. Franklin Inst.* **283**, 300–307 (1967).
8. D. F. Hays and H. N. Curd, Heat conduction in solids: Temperature dependent thermal conductivity, *Int. J. Heat Mass Transfer* **11**, 285 (1967).
9. I. J. Kumar and H. N. Narang, A boundary layer method in porous body heat and mass transfer, *Int. J. Heat Mass Transfer* **10**, 1095 (1967).
10. J. W. L. Rozenshtok, Application of boundary layer theory to the solution of problems with coupled heat and mass transfer. *J. Engng Phys.* (Translation from Russian by Faraday Press) **8**, 483 (1965).
11. A. V. Luikov and Yu. A. Mikhailov, *Theory of Heat and Mass Transfer*, p. 110. Israel Programme of Scientific Translations, Jerusalem (1965).
12. A. D. Booth, *Numerical Methods*, Chapter 9. Academic Press, New York (1955).

UNE FORMULATION VARIATIONNELLE GÉNÉRALISÉE DU TRANSFERT MASSIQUE ET THERMIQUE NON-LINÉAIRE DANS UN MILIEU POREUX

Résumé—Une formulation variationnelle basée sur la méthode de "potentiel local" est présentée pour le transfert thermique et massique non linéaire durant le séchage d'un corps humide. On obtient une fonctionnelle à partir de laquelle les équations du bilan décrivant le processus ressemblent à celles d'Euler–Lagrange dans le sens étendu de Prigogine et Glansdorff. La méthode a été appliquée au problème du transfert de chaleur et de masse pour une plaque infinie avec des diffusivités thermiques et massiques sous des conditions limites de première espèce.

EIN ERWEITERTES VARIATIONSPRINZIP FÜR INSTATIONÄREN WÄRME- UND STOFFTRANSPORT IM PORÖSEN MEDIUM

Zusammenfassung—Ein Variationsprinzip, abgeleitet aus der Methode der "lokalen Potentiale", wird am Beispiel des instationären Wärme- und Stofftransports während des Austrocknungsvorgangs in einem feuchten Körper erläutert. Eine Funktion wird abgeleitet, aus der die Gleichgewichtsbeziehungen des Vorgangs als Euler-Lagrange Gleichungen in der von Prigogine and Glansdorff erweiterten Form folgen. Das Verfahren wird auf das Problem der Wärme- und Stoffübertragung in einer unendlich ausgedehnten Platte, mit Randbedingungen 1. Art für die Temperatur und die Konzentration, angewandt.

ОБОБЩЕННЫЙ ВАРИАЦИОННЫЙ ПРИНЦИП ДЛЯ НЕЛИНЕЙНОГО ТЕПЛОИ МАССООБМЕНА В ПОРИСТОЙ СРЕДЕ

Аннотация—В работе представлен вариационный принцип, сформулированный на основании «локального потенциала», применяемый для решения нелинейной задачи- и массообмена в процессе сушки влажного пористого тела. Получен функционал, из которого следует уравнения баланса в виде уравнений Эйлера- Лагранжа, обощенных Пригожиным и Глансдорфом. Метод применяется к решению задачи о тепло- и массообмене на бесконечной пластине при температуре и влажности, зависимых от температуропроводности и массопроводности, при граничных условиях первого рода.